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SOME EVALUATIONS OF WEAK INSTABILITY FOR COLLIDING PROTON BEAMS

by

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I would like to draw attention to the possible existence of a very weak instability during the interaction of colliding proton beams. The assumed instability is of a dynamic nature and is, in fact, related to the movement of a single particle in the field of the colliding bunch (the so-called strong-weak interaction /1/). In the papers by Kolmogorov, Arnold and Moser /2/ it was shown that in a non-linear oscillation system, the majority of the invariant tori (integral surfaces) are conserved when subjected to a perturbation, if the latter is not too great. This results in perpetual stability in the case of one-dimensional non-autonomous or two-dimensional autonomous systems(*) when the non-conserved (resonant) tori are separated by the conserved tori. This stability was demonstrated in Laslett's numerical experiments /3/ with a non-linear transformation. In accordance with the information given in /4/ the experimental boundary of perpetual stability is determined approximately by the overlapping of the resonances.

In a multi-dimensional system the invariant tori do not separate phase space and, as a result, the unstable trajectory is no longer restricted. The concrete mechanism of instability, discovered by Arnold /5/ is related to diffusion along resonance surfaces intersecting in a multi-dimensional system, or more accurately, along stochastic layers in the region of the disrupted separatrices of the non-linear resonances. Hereafter, I will refer to this process as "Arnold diffusion".

(*) Hereafter, both cases will be called one-dimensional.

The first experimental indication of the existence of this phenomenon was apparently obtained in reference /6/ by means of a numerical calculation of the trajectories of particles in strong focusing non-linear field of an accelerator. Figure 1, which was taken from this work, shows the variation in oscillation amplitude along both axes as a result of disturbance. The very prolonged "latent" period of development of the instability, which apparently corresponds to slow diffusion along the stochastic layer (the upper curves in figure 1), is characteristic. Consequently, it is not at all clear whether, for example, the movement represented by the lower curves of figure 1 is in fact stable.

Another example of Arnold diffusion was observed in numerical experiments with a very simple model of a two-dimensional non-linear non-autonomous oscillator /7/. Figure 2, which was taken from the latter work, shows the variation with time of the product of the oscillation amplitudes over both degrees of freedom. Again, the instability develops very slowly at the beginning, and only at the end does it grow fast.

Finally, there are also certain preliminary data from "real" experiments concerning the motion of electrons in a magnetic trap /8,9/. Figure 3, taken from paper /9/ shows the family of curves characterising the dependence of the average life of an electron in the trap on the value of the magnetic field. A feature of these curves is the sharp decrease in the life of the particle by about one order of magnitude at a certain critical value of the magnetic field and the subsequent levelling out into a lower plateau. The interpretation of the results of a "real" experiment is a much more complicated matter than for a numerical experiment, because, in the first case, many important parameters and characteristics of the process remain unknown. Nevertheless, we may assume that, in this case too, Arnold diffusion takes place owing to weak azimuthal inhomogeneities of the field ($\sim 10^{-3}$), which makes the system multi-dimensional in the sense of the above-mentioned definition.

Arnold diffusion may play a part in storage rings, particularly proton storage rings, and still more so in antiproton storage rings, since in natural conditions there is a complete absence of any damping of the oscillations, and a beam life time of a few hours /10/ is necessary. Budker proposed introducing artificial damping, using an accompanying electron beam /11/; in this case everything depends on the damping time achieved.

The criterion of stochasticity by resonance overlapping /12/ makes it possible to evaluate the width of the stochastic layer /4/ and the rate of Arnold diffusion.

Let us consider the interaction of a single particle with a colliding bunch in a model similar to that of /13/. The small parameter of the problem is the relative shift in betratron oscillation frequency due to the colliding bunch : $\varepsilon = \Delta\nu/\nu$. In accordance with /3/, the coefficient of non-linearity $\alpha = \left|\frac{I}{\nu}\cdot\frac{d\nu}{dI}\right| \sim \varepsilon$, where I is the momentum, is of the same order of magnitude.

The condition of resonance is of the form :

$$n_1\nu_1 + n_2\nu_2 + \bar{m}\nu_0 = 0 \tag{1}$$

where n_1, n_2 and $\bar{m}$ are integers; and ν_0 is the frequency of the external disturbance which we shall consider as being δ like (any $\bar{m}$). We shall take the amplitude of the perturbing harmonic in the form :

$$\varepsilon_n \sim \varepsilon \cdot e^{-n/n_0} \tag{2}$$

where the parameter n_0 depends on the shape of the beam and the oscillation amplitude a. In particular for the Gaussian shape, $n_0 \sim a/r_0$, where r_0 is the beam radius. Let us now introduce a dimensionless coupling parameter between the degrees of freedom of β^2; in certain cases this parameter may be very small /10/;

In these conditions, the width of the external resonances : $\Omega_n \sim \sqrt{\varepsilon \cdot \varepsilon_n} \cdot \nu$, and the width of the coupling resonances $\Omega_n \sim \sqrt{\varepsilon \cdot \varepsilon_n} \cdot \beta \nu$ /12/. By using the results in paper /7/ it is easy to show that the average distance between the resonances up to the n-harmonic inclusively is given by the evaluation:

$$\Delta_n \sim \frac{\nu_o}{(2n)^3} \quad (3)$$

where it is assumed that $\nu_1 \sim \nu_2$.

The boundary of stochasticity is determined by the overlapping of the basic resonances ($n \leq n_o$); we obtain :

$$\varepsilon_s \sim (\nu_o/\nu) \cdot (2n_o)^{-3} \cdot \beta^{-1} \quad (4)$$

The numerical experiments of paper 1 give $(\varepsilon\nu)_s \approx 1/20$, and experiments with an electron storage ring give a value of $(\varepsilon\nu)_s \approx 1/40$

When $\varepsilon < \varepsilon_s$ the stochasticity is limited by a narrow strip in the neighourhood of the separatrix of the resonances, the width of which is $\delta_n \sim e^{-\omega_1/\Omega_n}$ /4/, where ω_1 is the minimum frequency of the separatrix - disrupting disturbance on the side of the nearest neighbouring resonance. Using (1) and (3) we obtain :

$$\omega_1 \sim \Delta_{n_1} \cdot n_1 \sim \frac{\nu_o}{2(2n_1)^2} \quad (5)$$

The harmonic of the disrupting resonance n_1 may be greater than n. Let us choose it from the condition : $\omega_1 \sim \Omega_n$ .We find :

$$n_1 \sim e^{\frac{n}{4n_o}} \sqrt{\frac{\nu_o}{8\varepsilon\nu\beta}} \sim e^{\frac{n}{4n_o}} \sqrt{\frac{\varepsilon_s}{\varepsilon}} \quad (6)$$

In the last evaluation the relation (3) was used and $n_o \sim 1$ was inserted. The condition $\omega_1 \sim \Omega_n$ is not, in fact, optimum. The optimum n_1 is found to be

$$n_1 \sim \sqrt[3]{\frac{\varepsilon_s}{\varepsilon}} \cdot e^{\frac{n}{6n_o}} \cdot n_o \quad (7)$$

It can be shown that the coefficient of Arnold diffusion is proportional to the value $(\delta_n^2 \beta^2 \varepsilon_{n_1})^2$. Consequently, as a rough estimate we can write :

$$\mathcal{D} \sim \mathcal{D}_o \beta^3 \exp\left(-3\sqrt[3]{\frac{\varepsilon_s}{\varepsilon}}\, e^{\frac{n}{6n_o}}\right) \tag{8}$$

where $\mathcal{D}_o \sim 2\pi I^2 \varepsilon \nu$ is the diffusion coefficient of strong stochasticity, occuring when the basic resonances overlap, i.e. when $\varepsilon > \varepsilon_s$.

When the proton life in the storage ring is very long, the ratio D_o/D may be very large, and consequently the fairly high harmonics are significant, namely :

$$n_A \approx 6n_o \ln\left(\sqrt[3]{\gamma\beta}\, \ln\sqrt[3]{\frac{\mathcal{D}_o\beta^3}{\mathcal{D}}}\right) \tag{9}$$

where $\gamma = \varepsilon/\varepsilon_s\beta$. The lower boundary of γ, when Arnold diffusion clearly does not play any part, is determined from the condition $n_A = 0$, or :

$$\beta\gamma_o = \left(\ln\sqrt[3]{\frac{\mathcal{D}_o\beta^3}{\mathcal{D}}}\right)^{-3} \tag{10}$$

In fact, this boundary is somewhat higher, since for small n_A the distance between the resonance is great and their action is ineffective. It is necessary for the distance between the resonances to be at least of the order of magnitude of the phase shift : $\Delta_A \lesssim \varepsilon\nu$ i.e. it must be (3) $(n_A/n_o) \gtrsim \gamma^{-1/3}$. In combination with (9) we obtain the equation for determination of the threshold γ :

$$6\gamma^{1/3} \ln\left(\sqrt[3]{\gamma\beta}\, \ln B\right) \approx 1\,;\quad B = \sqrt[3]{\frac{\mathcal{D}_o\beta^3}{\mathcal{D}}} \tag{11}$$

Assuming that, as a first approximation, $\gamma = 1/3$, we find :

$$\gamma_1 \approx \left[6 \ln \left(\frac{e_n B}{\sqrt{32}} \right) \right]^{-3} \qquad (12)$$

The Arnold diffusion occurs only for special initial conditions inside the stochastic layer. However, when there is additional diffusion, which may be due to scattering on the residual gas or to other fluctuations in the accelerator, the life may be reduced in the relation

$$k \sim \left(\Delta_A / \varepsilon \nu \right)^{-2} \sim \left(\gamma / \gamma_1 \right)^{2} \qquad (13)$$

Apparently, it was this very effect which was observed in paper (9) (see figure 3).

The very rough estimates mentioned above can be made more accurate for specific arrangements, by means of a numerical experiment. In accordance with paper /7/, it is sufficient to investigate the local stability of motion for relatively few revolutions.

I take this opportunity of expressing my sincere thanks to Messrs. Keil, Laslett, Sessler and Skrinskij for contributing to the interesting discussions.

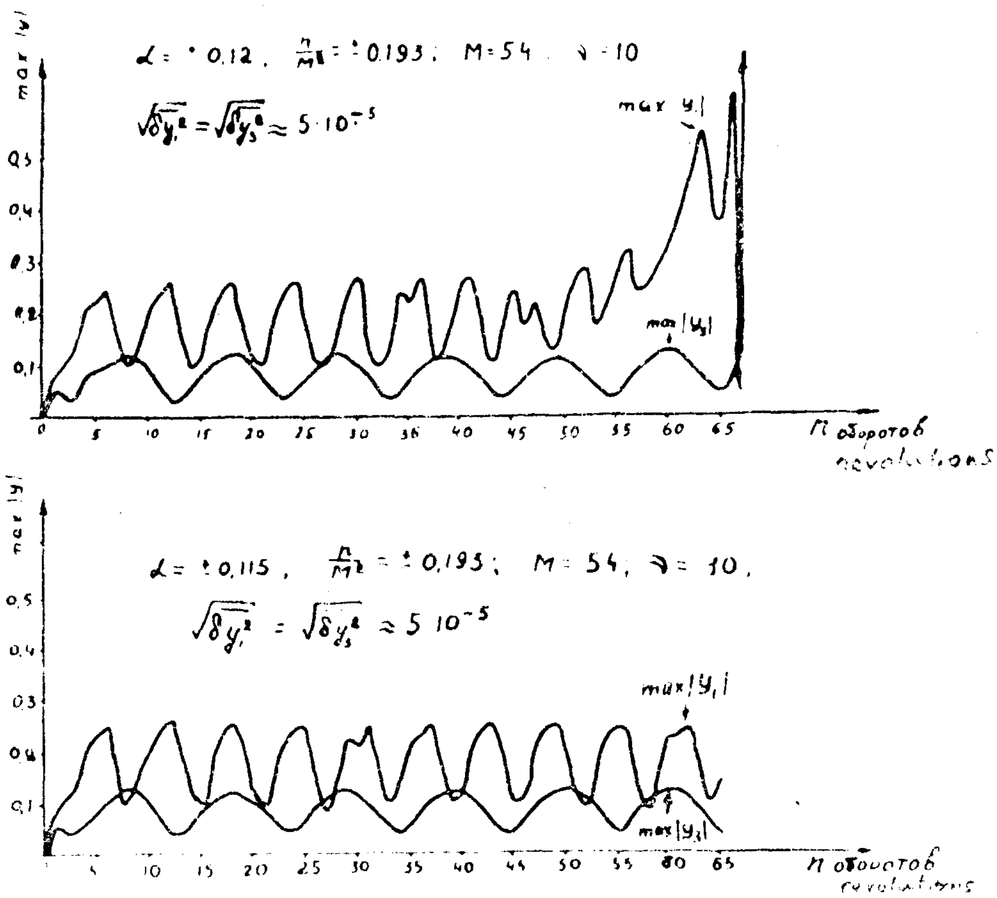

Figure 1.

Weak instability of the trajectories in a non-linear accelerator /6/ : α = parameter of non-linearity; ν = number of oscillations per revolution; $\bar{m}$ax /y/ = oscillation amplitudes.

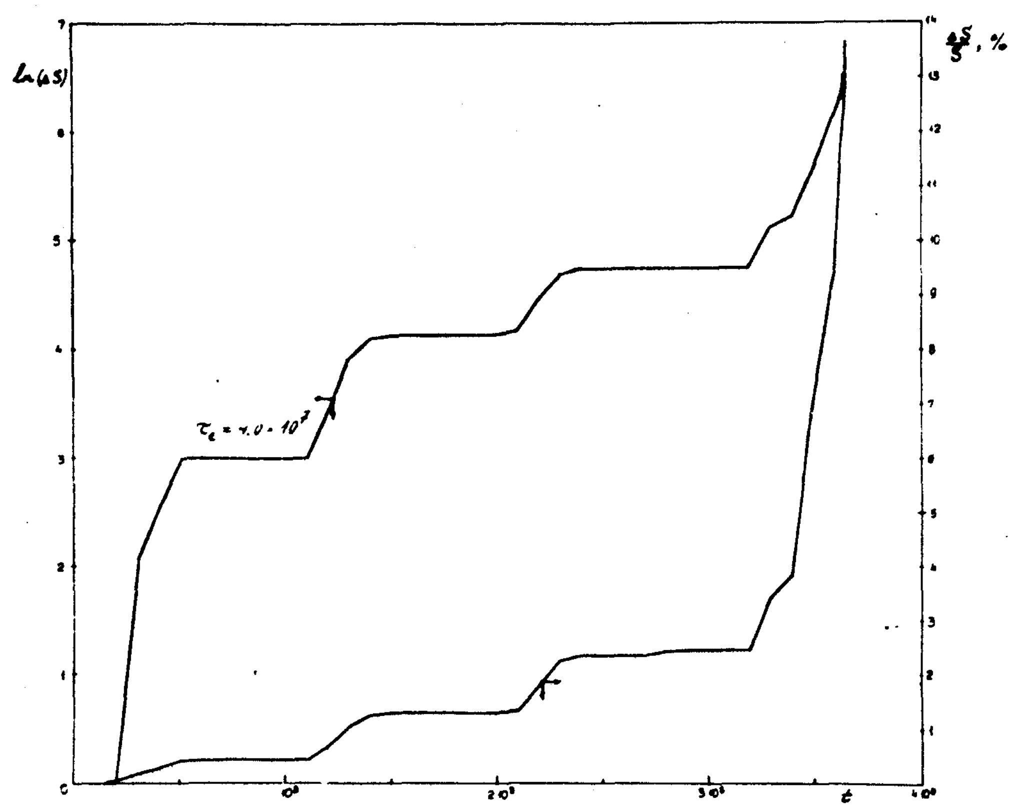

Figure 2.

Weak instability of a very simple non-linear multidimensional transformation /7/ : t time (number of steps); S = product of oscillation amplitudes over two degrees of freedom.

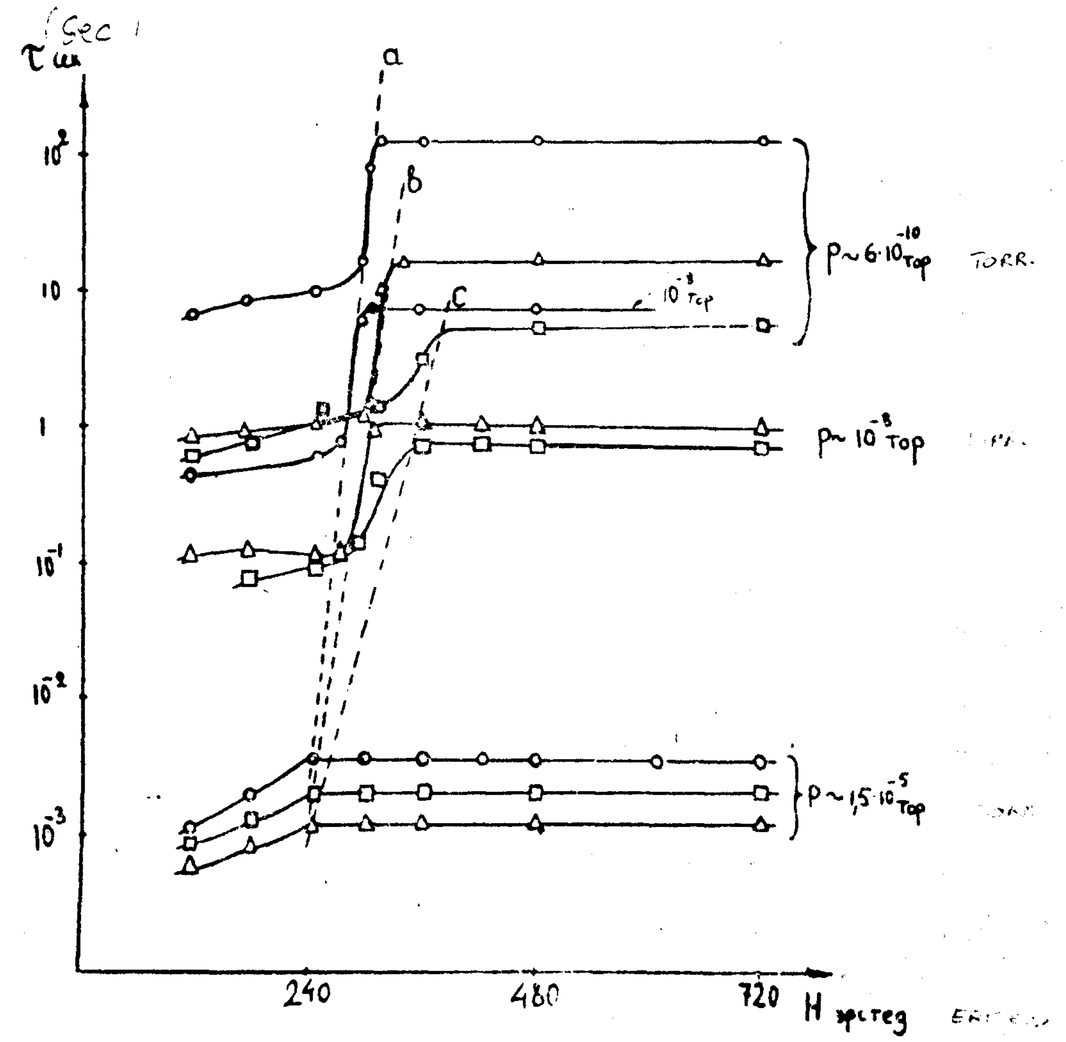

Figure 3.

Weak electron instability in a magnetic trap /9/ :
H = magnetic field of the trap; τ = average retention time of electron;
p = residual gas pressure.

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